OPTIMUM SHAPE OF RADIATION-COOLED RING FINS

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Optimum geometry is considered for axially symmetric radiation-cooled ring fins. The underlying body is a long cylinder or prism with the fins arranged along it, and the problem can be considered in two dimensions, as in the case of a single fin [1–6].

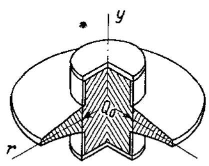

Fig. 1

Fins with shapes in certain classes have been considered [1–5] in relation to optimal size, e.g., rectangular [1, 2], triangular [3], power-law [4], and trapezoidal [5].

The variational problem has been considered [6] for the absolutely optimal shape for a planar fin with a given thickness at the edge. It is found that this shape initially coincides with the power-law form and then grades into a constant-thickness profile. The problem has thus been solved completely in the two-dimensional formulation.

A fin of the form shown in Fig. 1 may be used if the body is fairly short cylinder. Partial studies have been made on such fins, e.g., performance of a fin of constant thickness [7], the weight-optimum size of such a fin [8], and a system of fins of constant width (rectangular cross-section) [9].

Here we consider the optimal form of an axially symmetric single ring fin whose thickness varies in accordance with a certain class of function.

1. Formulation. Consider a cylindrical body of radius r_0 cooled by a fin of variable thickness (Fig. 1), into which the heat enters uniformly through the base. The external space is a vacuum at absolute zero.

If there are no heat sources within the fin and the material is isotropic, we can use an r-y coordinate system, in which r is reckoned from the axis of rotation and y is reckoned from the plane of symmetry of the fin.

Consider the steady state, with a constant temperature at the base of the fin, which is thin and varies slowly in thickness, so that the heat flux in the y direction may be neglected relative to the radial heat flux.

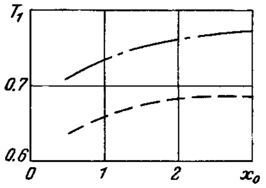

Fig. 2

The equation of heat transfer along the fin is

$$Q = -4\pi r y \lambda dT \,/\, dr, \tag{1.1}$$

and Stefan's law gives

$$dQ \,/\, dr = -4\pi r \sigma \varepsilon T^4, \tag{1.2}$$

where y is half the thickness of the fin, Q is heat flux along the fin, λ is thermal conductivity, T is temperature, σ is Stefan's constant, and ε is the emissivity (ε and σ are taken as constants).

The optimum shape for a fin of minimum weight must provide a minimum in

$$V = 2\int_{r_0}^{r_1} 2\pi r y \, dr \tag{1.3}$$

for given Q_0 and T_0; subscript 0 relates to quantities taken at the base of the fin, while subscript 1 relates to quantities at the outside edge.

Introducing the new variable $x = r^2$, (1.1)–(1.3) become

$$Q = -8\pi\lambda x y \frac{dT}{dx}, \quad \frac{dQ}{dx} = -2\pi\sigma\varepsilon T^4, \quad V = 2\int_{x_0}^{x_1} \pi y \, dx. \tag{1.4}$$

We introduce the following dimensionless quantities:

$$Q^\circ = \frac{Q}{Q_0}, \quad T^\circ = \frac{T}{T_0}, \quad x^\circ = \frac{2\pi\sigma\varepsilon T_0^4 x}{Q_0},$$

$$y^\circ = \frac{8\pi\lambda T_0 y}{Q_0}, \quad V^\circ = \frac{16\pi\lambda\sigma\varepsilon T_0^5 V}{Q_0^2}. \tag{1.5}$$

Then

$$Q = -xy\frac{dT}{dx}, \quad \frac{dQ}{dx} = -T^4, \quad V = 2\int_{x_0}^{x_1} y dx. \tag{1.6}$$

The superscripts to dimensionless quantities are omitted here and subsequently.

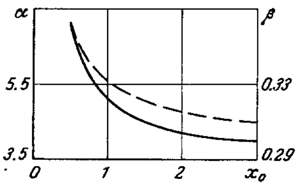

Fig. 3

We take the boundary conditions in the form

$$x = x_0, \ Q = 1, \ T = 1; \ x = x_1, \ Q = 0, \ T = T_1, \tag{1.7}$$

in which x_1 and T_1 are not known in advance.

The problem may now be formulated mathematically as that of finding the y(x), Q(x), and T(x), together with x_1 and T_1, that give a minimum in (1.6) and satisfy (1.7).

2. Solution. We specify a certain form for y(x). We consider three cases:

a) The case in which

$$y = y_0 x_0 \,/\, x, \tag{2.1}$$

where y_0 is an unknown constant to be found from the condition for optional V, which here takes the form

$$V = 2x_0 y_0 \ln x_1 \,/\, x_0. \tag{2.2}$$

From the first two parts of (1.6) we have

$$d^2T \,/\, dx^2 = T^4 \,/\, x_0 y_0 \,. \tag{2.3}$$

An equation of this type has been derived [2] for the planar case for a fin of rectangular cross-section. We integrate in (2.3) and use the third and fourth boundary conditions in (1.7) to get

$$Q = \sqrt{{}^2/_5\, x_0 y_0 (T^5 - T_1^5)},$$

$$B\left(^3/_{10},\ ^1/_2\right) - B_\theta\left(^3/_{10},\ ^1/_2\right) = \sqrt{10/x_0y_0}\, T_1^{3/2}(x_1 - x), \tag{2.4}$$

which contain the unknown parameters T_1, x_1, and y_0; here $\theta = (T_1/T)^5$, while B and B_θ are the complete and incomplete β functions.

The other boundary conditions allow us to express y_0 and x_0 in terms of T_1:

$$y_0 = \frac{5}{2}\frac{1}{x_0(1 - T_1^5)},$$

$$x_1 - x_0 = \frac{B\left(^3/_{10},\ ^1/_2\right) - B_\gamma\left(^3/_{10},\ ^1/_2\right)}{2(1 - T_1^5)^{1/2}T_1^{3/2}}, \text{ in which } m = T_1^5. \tag{2.5}$$

Substitution of (2.5) into (2.2) leads us to minimize

$$V(T_1, x_1) = \frac{5}{1 - T_1^5}\ln\frac{x_1}{x_0} \tag{2.6}$$

subject to the condition of (2.5).

The problem has been solved for x_0 from 0.5 to 3 by steps $\Delta x_0 = 0.5$; the dot-dash lines in Figs. 2 and 4–6 represent the optimum T_1, $x_1 - x_0$, y_0, and V as functions of x_0.

Consider the solution as $x_0 \to \infty$. We see from Fig. 4 that $x_1 - x_0$ decreases as x_0 increases, while x_1 increases, so $(x_1 - x_0)/x_1 = \delta$ becomes small for x_0 sufficiently large. We expand (2.6) as a power series in δ:

$$V = -\frac{5}{1 - T_1^5}\ln(1 - \delta) =$$

$$= \frac{5}{1 - T_1^5}\left(\delta + \frac{\delta^2}{2} + \ldots + \frac{\delta^n}{n} + \ldots\right).$$

We retain only the early terms to get

$$V \approx \frac{5}{1 - T_1^5}\frac{x_1 - x_0}{x_0}, \qquad \left(x_1 = \frac{x_0}{1 - \delta} \approx x_0 + O(\delta)\right). \tag{2.7}$$

It is readily shown [2] that we have to find a minimum in

$$F = \frac{5L}{1 - T_1^5} \quad \text{for } L = \frac{B\left(^3/_{10},\ ^1/_2\right) - B_\gamma\left(^3/_{10},\ ^1/_2\right)}{2(1 - T_1^5)^{1/2}T_1^{3/2}} \tag{2.8}$$

in order to determine the optimum planar fins of rectangular profile, for which the symbols of [6] are used.

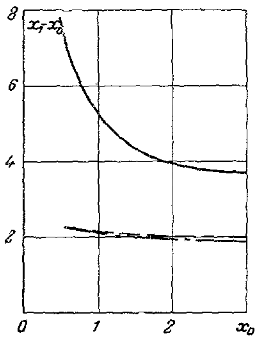

Fig. 4

Comparison of (2.7.1) with (2.8.1) and (2.5.2), with (2.8.2) shows that the functions in the present case differ from those of [2] only in the factor $1/x_0$ for x_0 sufficiently large, and this factor is not dependent on T_1. Then $T_{1\mathrm{opt}}$ for the present fin tends to 0.799 as x_0 increases, which is the value for a rectangular fin.

b) The case in which

$$y(x) = y_0\frac{x_0}{x}\frac{x_1 - x}{x_1 - x_0}. \tag{2.9}$$

Then

$$V = 2x_0y_0\left(\frac{x_1}{x_1 - x_0}\ln\frac{x_1}{x_0} - 1\right), \tag{2.10}$$

while (1.6) and (1.7) give

$$\frac{d}{dx}\left[\frac{x_1 - x}{x_1 - x_0}\frac{dT}{dx}\right] = \frac{1}{x_0y_0}T^4, \tag{2.11}$$

$$x = x_0, \quad \frac{dT}{dx} = -\frac{1}{x_0y_0}, \quad T = 1; \qquad x = x_1, \quad T = T_1. \tag{2.12}$$

Three of the conditions in (1.7) are obeyed exactly, since the fin has a sharp edge ($x = x_1$, $y = 0$).

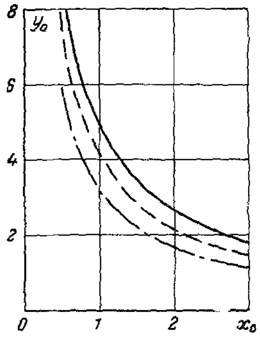

Fig. 5

An equation of the type of (2.11) has been derived for a triangular fin [4], and the method of [4] will be applied.

We introduce a new unknown function v and a new independent variable u, which are related to the old ones by

$$T = \gamma v(u), \quad u = \beta\frac{x_1 - x}{x_1 - x_0}, \qquad \left(\frac{\beta}{\gamma^3} = \frac{(x_1 - x_0)^2}{x_0y_0}\right), \tag{2.13}$$

in which β and γ are constants to be determined. Then (2.11) and (2.12) become

$$\frac{d}{du}\left(u\frac{dv}{du}\right) = v^4, \tag{2.14}$$

$$u = \beta, \quad \frac{x_0y_0}{x_1 - x_0}\gamma\beta v'(\beta) = 1, \quad \gamma v(\beta) = 1;$$

$$u = 0, \quad \gamma v(0) = T_1, \tag{2.15}$$

in which a prime denotes the derivative with respect to u. We put γ as

$$\gamma = 1 / v(\beta). \tag{2.16}$$

Then (2.12) and (2.13) allow us to put y_0, x_1, and T_1 in terms of β:

$$y_0 = \frac{1}{x_0}\frac{v^5(\beta)}{\beta v'^2(\beta)} \qquad x_1 = x_0 + \frac{v^4(\beta)}{v'(\beta)}, \qquad T_1 = \frac{v(0)}{v(\beta)}. \tag{2.17}$$

In that case, the boundary conditions of (2.15) are obeyed exactly, no matter what the value of v(0). We put $v(0) = 1$; then $v'(0) = 1$, because the solution to (2.14) is regular at $u = 0$.

It has been shown [4] that the solution to (2.14) subject to $v(0) = v'(0) = 1$ may be put as the series

$$v = 1 + u + u^2 + 1.1111u^3 + 1.2778u^4 + 1.4978u^5 +$$
$$+ 1.7775u^6 + 2.1279u^7 + 2.5638u^8 + \ldots,$$

whose radius of convergence is not less than 0.5. We then have to find the β such that

$$V = 2\frac{v(\beta)}{v'(\beta)}\left[\left(x_0 + \frac{v^4(\beta)}{v'(\beta)}\right)\ln\frac{x_0 + v^4(\beta)/v'(\beta)}{x_0} - \frac{v^4(\beta)}{v'(\beta)}\right] \tag{2.18}$$

is minimal.

The v and v' of (2.18) are

$$v(\beta) = 1 + \beta + \beta^2 + 1.1111\beta^3 + 1.2778\beta^4 + 1.4978\beta^5 +$$
$$+ 1.7775\beta^6 + 2.1279\beta^7 + 2.5638\beta^8 + \ldots,$$

$$v'(\beta) = 1 + 2\beta + 3.3333\beta^2 + 5.1112\beta^3 + 7.4890\beta^4 + \\ + 10.665\beta^5 + 14.895\beta^6 + 20.510\beta^7 + \ldots.$$

The computations were performed with a Razdan-2 computer; Fig. 3 shows the resulting optimum dependence of β on x_0. The dashed lines in Figs. 2 and 4-6 represent T_1, x_1, y_0, and V as functions of x_0.

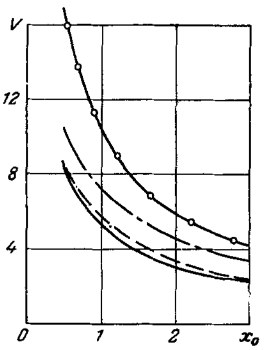

Fig. 6

As previously, the behavior of $x_1 - x_0$ is such that (2.10) for x_0 large may be expanded as a power series in δ:

$$V = 2\frac{v^5(\beta)}{\beta v'^2(\beta)}\left[\frac{\delta}{2} + \frac{\delta^2}{3} + \ldots + \frac{\delta^n}{n+1} + \ldots\right] \quad \left(\delta = \frac{x_1 - x_0}{x_1}\right). \tag{2.19}$$

We take only small quantities of the first order of smallness to get

$$V = \frac{1}{x_1}\frac{v^9(\beta)}{\beta v'^3(\beta)} \approx \frac{1}{x_0}\frac{v^9(\beta)}{\beta v'^3(\beta)}. \tag{2.20}$$

The optimization of a triangular fin has [4] been reduced to minimizing $v^9(\beta)/\beta v'^3(\beta)$ subject to the conditions of (2.18), while the resulting β_{opt} was 0.287, so we may say that the optimum β for the present case [fin of the form of (2.9)] will tend to 0.287 as $x_0 \to \infty$.

c) The case where

$$y = y_0 \frac{x_0}{x}\left(\frac{x_1 - x}{x_1 - x_0}\right)^\alpha, \tag{2.21}$$

in which y_0 and α are unknown parameters to be found (as well as x_1 and T_1) by optimization. The functional is

$$V = 2x_0 y_0 \int_{x_0}^{x_1} \frac{1}{x}\left(\frac{x_1 - x}{x_1 - x_0}\right)^\alpha dx, \tag{2.22}$$

while (1.6) becomes

$$\frac{d}{dx}\left[\left(\frac{x_1 - x}{x_1 - x_0}\right)^\alpha \frac{dT}{dx}\right] = \frac{1}{x_0 y_0} T^4, \tag{2.23}$$

which is readily integrated. The first and second conditions in (1.7) give the optimum T(x) and Q(x):

$$T = \left(\frac{x_1 - x}{x_1 - x_0}\right)^{1/3(\alpha - 2)}, \qquad Q = \left(\frac{x_1 - x}{x_1 - x_0}\right)^{1/3(4\alpha - 5)}. \tag{2.24}$$

Then the optimum value of T_1 is 0.

The other boundary conditions allow us to express all the unknown parameters in terms of α:

$$x_1 = x_0 + \frac{4\alpha - 5}{3}, \qquad y_0 = \frac{1}{x_0}\frac{4\alpha - 5}{\alpha - 2}. \tag{2.25}$$

It remains to determine the α that minimizes

$$V = 2\frac{4\alpha - 5}{\alpha - 2}\int_{x_0}^{x_1} \frac{1}{x}\left(\frac{x_1 - x}{x_1 - x_0}\right)^\alpha dx \tag{2.26}$$

subject to (2.25), it being clear from (2.24) that $\alpha > 2$ from physical considerations.

We introduce the new independent variable $t = x/x_1$ to transform (2.26) to

$$V = 2\frac{4\alpha - 5}{\alpha - 2}\frac{1}{(1 - t_0)^\alpha}\int_{t_0}^{1} \frac{1}{t}(1 - t)^\alpha dt. \tag{2.27}$$

To find $V(\alpha)$ we may expand $1/t$ as a power series in $(1 - t)$. Then the integrand is a uniformly convergent series, which integrates to

$$V = 2\frac{4\alpha - 5}{\alpha - 2}\left[\frac{1 - t_0}{\alpha + 1} + \frac{(1 - t_0)^2}{\alpha + 2} + \ldots + \frac{(1 - t_0)^n}{\alpha + n} + R_{n+1}\right]. \tag{2.28}$$

It is readily shown that

$$R_{n+1} < (1 - t_0)^{\alpha + n} \ln t_0.$$

The solid lines in Figs. 3-6 show the optimum α as derived with a Razdan-2 computer from the $V(\alpha)$ for various α.

The quantity $(1 - t_0) = (x_1 - x_0)/x_1 = \delta$ is small for x_0 sufficiently large, so (2.28) gives

$$V \approx 2\frac{4\alpha - 5}{(\alpha - 2)(\alpha + 1)}\frac{x_1 - x_0}{x_1} \approx 2\frac{4\alpha - 5}{(\alpha - 2)(\alpha + 1)}\frac{x_1 - x_0}{x_0}.$$

Then (2.25) gives

$$V = \frac{2}{3}\frac{(4\alpha - 5)^2}{(\alpha + 1)(\alpha - 2)}\frac{1}{x_0}. \tag{2.29}$$

It is then readily found that $\alpha_{opt} = 3.5$, which agrees with the optimal degree found for a power-law fin [4].

3. Conclusions. Figure 6 shows that V decreases as x_0 increases, as is to be expected, since the heat flux per unit length of root at a fixed Q_0 then decreases.

Figure 6 also shows that the fin of (2.21) is the best of those considered. Such fins have very sharp edges, so it is better to consider a fin of the form of (2.9), which results in not more than 7% increase in weight for the x_0 considered. A fin of the type of (2.1) increases the weight by not less than 25%.

The line with circles in Fig. 6 is the optimum $V(x_0)$ for a ring fin of constant thickness (rectangular cross-section), as derived from [8]. It is clear that such a fin increases the weight by over 90% relative to the optimal forms.

REFERENCES

1. I. G. Bartas and W. H. Sellers, "Radiation fin effectiveness," Trans. ASME. J. Heat Transfer, vol. 82, no. 1, 1960.

2. Y. L. Chen, "On minimum-weight rectangular radiating fins," J. ASS, vol. 27, no. 11, 1960.

3. E. N. Nilson and R. Curry, "The minimum-weight straight fin of triangular profile radiating to space," J. ASS, vol. 27, no. 2, 1960.

4. J. E. Wilkins, Jr., "Minimizing the mass of thin radiating fins," J. ASS, vol. 27, no. 2, 1960.

5. J. E. Wilkins, Jr., "Minimum mass thin fins for space radiators," Proc. 1960 Heat Transfer and Fluid Mechanics Institute, Stanford University.

6. G. L. Grodzovskii, "Optimum form of radiation-cooled conducting fins," Izv. AN SSSR, OTM, Energetika i Automatika, no. 6, 1962. (See also Proc. XII Internat. Astronaut. Congr., 1961, Vine, 1962, Astronautica acta, no. 4, 1962).

7. R. L. Chambers and E. V. Somers, "Radiation fin efficiency for one-dimensional heat flow in a circular fin," Trans. of the ASME, C. (J. Heat Transfer), vol. 81, p. 327–329, 1959.

8. D. B. Mackay, "Transmission de chaleur par rayonnement des surfaces ailetées dans un espace environnant et son optimisation," Journees internat. transmiss. chaleur, 19–24 juin 1961, Paris, vol. 2, p. 629–663, 1962.

9. E. M. Sparrow, G. B. Miller, and V. K. Jonsson, "Radiating effectiveness of annular-finned space radiators, including mutual irradiation between radiator elements," JASS, vol. 29, no. 11, 1962.

18 February 1967 Moscow, Kiev